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Symplectic and Lagrange Stabilities of Open Whitney Umbrellas (G.Ishikawa 石川剛郎氏).

Let $f : X \to M$ be a map-germ to a symplectic manifold (M, ω). In the space V_f of vector fields along f, set $VI_f = \{v \in V_f \mid v^\sharp : X \to T^*M \text{ is isotropic }\}$, $CI_f = \{(df_t/dt)|_{t=0} \mid f_t^*\omega = f^*\omega, f_0 = f\}$. Then VI_f (resp. CI_f) is a vector subspace (resp. a cone) of V_f. In general, $CI_f \subset VI_f$. If corank of f is at most 1, then $CI_f = VI_f$. There are examples f of corank 2 with $CI_f \neq VI_f$.

Question 1: Is CI_f equal to VI_f, if f is symplectically stable?

Let ρ be a germ of closed 2-form on X. Set $V_\rho = \{\xi \in V_X \mid L_\xi \rho = 0\}$. The following naturally arises in the study of PDE $f^*\omega = \rho$:

Question 2: Can we define a natural algebraic structure on V_ρ?

To classify symplectically (Lagrange) stable map-germs, we are naturally led to the following:

Question 3: What is the symplectic counterpart of the K(contact)-equivalence?

安定写像の特異点の除去問題 (佐久間一浩氏).

1. F^2 を non-orientable closed surface とする。Smooth embedding $\phi : F^2 \to \mathbf{R}^4$ に対して、generic projection $\pi : \mathbf{R}^4 \to \mathbf{R}^3$ を考える。このとき、$\pi \circ \phi$ は、stable map であるが、ϕ の isotopy class を変えずに triple points を消去することが、出来るか。

2. M^4 を simply connected closed 4-manifold とする。M^4 が special generic map $f : M^4 \to \mathbf{R}^3, s.t. S(f)$: *connected* を許容するならば、M^4 は 4 次元球面に微分同相か。ただし、$S(f)$ は f の singular set を表し、いまの場合 2 次元球面に微分同相である。また、$M^4 \cong S^4$ のとき、$S(f)$ は smoothly unknotted か。

3. M^4 を closed orientable 4-manifold with odd euler characteristic とする。このとき、cusp を持たない安定写像 $f : M^4 \to \mathbf{R}^3$ は存在するか。

The Global Theory of Real Singularities (T.Ohmoto 大本 亨氏).

1. *Elimination problem of singularities of smooth mappings.* Consider elimination (via smooth homotopy) of isolated singularities $\Sigma(f)$ of smooth stable mappings $f : N^n \to P^p$ (i.e., the case that $\operatorname{codim}\Sigma = n$). We have known some results of A.DuPlessis and Y.Ando ([1],[2]): Roughly speaking, for some good classes Σ (of Thom-Boardman singularities or contact singularity), if the Thom polynomial of Σ for f vanishes, then there is a smooth homotopy $F : N \times [0,1] \to P$ s.t. $F_0 = f$ and F_1 is a smooth stable mapping without Σ-type singularity (i.e. $\Sigma(F_1) =$). Namely, Thom polynomials are considered as obstruction classes to elimination via smooth homotopy. This is showed by "h-principle technique". But I don't know any similar results for multiple-singularities. Maybe, for triple points of stable mappings from a closed surface into $\mathbb{R}^3$, there must have been known some results... but I don't know anything even about this case.

2. *On characteristic numbers of images of stable mappings.* There is a formula of Euler characteristic of images of stable mappings f from a closed surface N into a

3-manifold P, due to Izumiya and Marar [3] : $\chi(f(N)) = \chi(N) + T(f) + C(f)/2$ where $T(f)$ means the number of triple points and $C(f)/2$ the number of pairs of Whitney umbrellas. On the other hand, there is a work of Vassiliev [4] on a differential complex associated to multi-singularities of stable mappings. Then, I'd like to propose a problem (cf [5]) : How should we generalize the formula of Izumiya and Marar in terms of Vassiliev's differential complex ?

3. *Reidemeister moves for 2-knots.* To recoginize objects in higher dimensional spaces, it is natural to look at images of it under projections to some lower dimensional spaces. For instance, in the classical knot theory, knot diagrams and Reidmeister moves are usuful to understand knots in 3-space. Now consider 2-sphere smoothly embedded in $\mathbb{R}^4$, and images of it via a coodinate projection $\mathbb{R}^4 \to \mathbb{R}^3$. Generically, its image is an immersed surface with normal crossings except at isolated singularities of Whitney umbrellas. On the other hand, using D. Mond's classfications of (multi) map-germs $(\mathbb{R}^2, S) \to (\mathbb{R}^3, 0)$ [6], we can know one parameter generic local bifurcations of images via the projection. Then, it naturally arizes some problems : How should we describe "Reidemeister moves" of 2-sphere embedded in $\mathbb{R}^4$ via a fixed projection to $\mathbb{R}^3$? And futhermore, can we define some numerical invariants (like as Vassiliev knot invariants) for 2-knots in $\mathbb{R}^4$ in terms of generic local bifurcations of its images via the projection ?

4. *Desingularizations of Thom-Boardman singularities.* Let $\Sigma^I := \Sigma^I(N, P)$ be the Thom Boadman strata in the jet space $J := J^k(N, P)$, where N and P are C^∞-manifolds (or complex analytic manifolds) and k is a sufficiently large number. We want a method to construct a "desingularization" of $\overline{\Sigma^I}$, that is, a triple of a fibre bundle $p : F \to J$, a vector bundle $E \to F$, a section $s : F \to E$ transverse to the zero sections such that the projection p sends the pull back $\tilde{\Sigma^I}$ ($:= s^{-1}(0)$) onto the closure $\overline{\Sigma^I}$ diffeomorphically off $\tilde{\Sigma^I} \cap p^{-1}(\overline{\Sigma^I} - \Sigma^I)$. I don't know even if such desingularizations always exist for any Σ^I.

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Topological triviality of map-germs (S.Koike 小池敏司氏).

I.Nakai showed topological moduli appear even in the families of polynomial map-germs (Topology 23 (1984), 45-66). His examples, of course, don't become Thom-maps for any locally finite stratification. Then I want to know when a family of non-Thom-maps becomes topologically trivial. Formulate any condition which implies the topological triviality and show it.

Let $f:(\mathbf{R}^n,0)\to(\mathbf{R}^n,0)$ be a polynomial mapping. We call f *an algebraic modification mapping*, if $f|_{\mathbf{R}^n-S(f)}:\mathbf{R}^n-S(f)\to\mathbf{R}^n-D(f)$ is bijective. Here $S(f)$ and $D(f)$ denote the singular points set and the discriminant set of f, respectively. Concerning the above problem, we give the following problems:

PROBLEM. *Construct a family of algebraic modification mappings in which local topological moduli appear.*

PROBLEM. *Let $f_t:(\mathbf{R}^3,0)\to(\mathbf{R}^3,0)$ $(t\in J)$ be an algebraic modification mapping. Then is the number of topological types of $\{f_t\}_{t\in J}$ finite?*

微分方程式論に関連する問題.（泉屋周一氏）.

1. $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_r)$ とおくとき、常微分方程式系

$$\begin{cases} \dot{x} &= f(x,y), \\ 0 &= g(x,y) \end{cases}$$

の分類理論をつくること。ただし、$f=(f_1,\ldots,f_n), g=(g_1,\ldots,g_r)$.

この問題は故池上氏の問題と言って良い。上記のシステムの特異摂動を考えると、

$$\begin{cases} \dot{x} &= f(x,y), \\ \varepsilon\dot{y} &= g(x,y) \end{cases}$$

となり、最初のシステムは後のシステムの $\varepsilon\to 0$ のときの極限と考えられる。この時、$\varepsilon\dot{y}=g(x,y)$ を$\dot{y}=\frac{1}{\varepsilon}g(x,y)$ と解釈すると、その極限であるシステムは y方向へ無限に早いスピードで動く力学系と考えることができる。従って、この場合 $\pi:\mathbb{R}^n\times\mathbb{R}^r\to\mathbb{R}^n$ を保つような微分同相等で分類するのが自然である。$n=2, r=1$ の場合は Arnold, Davidov がコントロール理論や接触幾何と関連させて分類を試みている。しかし、$r=2$ の場合は早い方向が２次元なので、おもしろい現象が解る可能性がある。

2. $J^2(\mathbb{R},\mathbb{R})$ の余次元１の部分多様体の様々な性質を調べよ。このような、部分多様体は２階の常微分方程式 $F(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2})=0$ と理解できる。２階の接触幾何の部分多様体論の足がかりとして、特に特異解、方程式の特異点、解の特異点と関連した性質を研究することはまったく未知と思える分野への足がかりとなる可能性がある。

3. ミンコフスキー空間の中の超曲面のガウス写像の特異点を研究せよ。ユークリッド空間の場合とどの様に違うか？調和写像との関連等について等、しかし、私は今の所なんの情報もありません。

4. $\mathbb{R}^3$の中の Caustics の Generic projection の研究。Caustics は３次元空間の中に出来るけれども、実際それを認識するときはスクリーン上に映し出すことに依るのでより詳しい状況を把握するためにはその projection の研究が必要となる。

5. 平均曲率の発展方程式に出てくる解曲面の特異点の分類。この問題は近年の、応用解析における主問題の一つである。いわゆる、粘性解の等高面としての存在定理及び一意性の定理などは解析的手法によって近年爆発的な発展をみているが、その特異点については特殊な場合しか解っていない。特異点理論のひとつの進むべき道として格好の題材と思われる。

写像芽の安定摂動の特異点の個数の勘定 (福井敏純).

結構次元 (nice range) (n,p) に対して、写像芽 $f:(\mathbf{C}^n,0)\to(\mathbf{C}^p,0)$ の安定摂動の特異点の個数を数える問題に関連して次の問題を提出したい。

1. ジェット空間内にある Thom-Boardman 多様体の Zariski 閉包を定義する自然なイデアルを見つけよ。またそのイデアルが定義する variety が Cohen-Macaulay であるような点を決定せよ。B.Morin 氏は "Calcul jacobian" (Ens 8.,1975) でこの問題を考え（たと思われ）イデアル Δ^I を定義し、Thom-Boardman 多様体 Σ^I はΔ^I の定義する variety Z_I の非特異点であることを示した。しかし一般には Z_I は Σ^I の Zariski 閉包より大きい。Σ^I の Zariski 閉包を定義するよいイデアルを見つける事ができれば写像芽の安定摂動の A_k, D_4, E_6, $I_{2,2}$, S_5 特異点等の個数を与える代数的な公式を Cohen-Macaulay の仮定の下に与える事ができる。

2. Thom-Boardman 多様体を調べるだけでは D_k $(k\geq 5)$, S_6, S_7, E_7, $I_{a,b}$ 等の特異点を特徴づけることはできない。従って写像芽の安定摂動に表れるそれらの特異点の個数について上と同様な事をするには次の問題を考える必要がある。ジェット空間内にある上記の contact orbits (K-orbits) の Zariski 閉包を定義する自然なイデアルを見つけよ。またそのイデアルが定義する variety が Cohen-Macaulay であるような点を決定せよ。

この問題 2 は大本氏によって福井に提出された問題である。

3. 上記で見つけたイデアルの定義する variety が Cohen-Macaulay でない点でこの variety と f の jet section との交点数を記述するできるだけ簡単な代数的な公式を見つけよ。

Problems from Partial Differential Equations

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1° Singularities of elementary solutions of hyperbolic equations with constant coefficients.

Let $P(\zeta)$ ($\zeta \in \mathbf{C}^n$) be a polynomial of degree m and $\vartheta \neq 0$ ($\vartheta \in \mathbf{R}^n$). $P(\zeta)$ is called to be hyperolic with respect to the direction ϑ if and only if 1) $P_m(\vartheta) \neq 0$ where P_m is the principal part of $P(\zeta)$ and 2) There exists $\gamma_0 > 0$ such that $P(\xi - i\gamma\vartheta) \neq 0$ for any $\xi \in \mathbf{R}^n$ and any $\gamma > \gamma_0$. An elementary solution $E(x)$ of $P(D)$ whose support is contained in $\{x \in \mathbf{R}^n; \langle x, \vartheta \rangle \geq 0\}$ is written as follows:

$$E(x) = \int_{R^n - i\gamma\theta} P(\zeta)^{-1} \exp(i\langle x,\zeta\rangle)\, d\zeta$$

Problem: To determine the singularities of E(x) exactly.

This problem has a long history. It is sure that topological properties of the algebraic surface $\{\zeta \in \mathbf{C}^n; P(\zeta)=0\}$ give great influence on E(x). Therefore our interest is to obtain some relations between the topological properties of $\{\zeta \in \mathbf{C}^n; P(\zeta)=0\}$ and the singularities of E(x). See [1].
At today's point, the best result would be [3].

2° Integration of algebraic functions of several complex variables.

Many problems in physical sciences are written as "Initial-boundary value problems for partial differential equations, especially for hyperbolic equations". Then the solutions are often expressed as the integral of algebraic functions of several complex variables. As the propagation of waves is mathematically translated as the prppagation of singularities, it becomes important to determine the singularities of functions written as the integral of algebraic functions of several complex variables. Then the algebraic singularities of the kernel cause various kinds of phenomena, for example "lateral waves" in elastic waves. Our problem is to get the relation between the singularities of the function and the singularities of the kernel. For example, at first we extend the kernel analytically, then we may arrive the integral on a kind of Riemann surface. Could we get the beautiful theory as [5] ?
As the survey lectures on these subjects, see [2] and [4].

I am afraid that I might write these things from my ignorance. I am very glad if you would kindly teach me something, especially geometric theory, on these subjects.

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